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On Multipliers Preserving the Classes of Functions with a Given Major of the Modulus of Continuity

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A necessary and sufficient condition is obtained for (C_ω, C_ω) multipliers in the case of a slowly decreasing majorant of the modulus of continuity $\omega(\delta)$. For the Lipschitz classes this result is equivalent to a theorem of A. Zygmund (*J. Math. Mech.* **8** (1959), 889–895). © 1991 Academic Press, Inc.

1. INTRODUCTION

Let X and Y be two classes of 2π-periodic integrable functions. We say that a two-way infinite sequence of complex numbers $\lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ is a multiplier from X into Y, and we write $\lambda \in (X, Y)$, if whenever

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

is the Fourier series of a function f in X, the series

$$\sum_{k=-\infty}^{\infty} \lambda_k c_k e^{ikx} \tag{2}$$

is the Fourier series of a function f_λ in Y.

Let C denote the class of 2π-periodic continuous functions with the norm

$$\|f\|_C = \max_{-\pi \leqslant x \leqslant \pi} |f(x)|$$

and L the class of 2π-periodic integrable functions with the norm

$$\|f\|_L = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\, dx.$$

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Let $\omega(\delta)$ be a given modulus of continuity and let C_ω denote the class of continuous functions, for the moduli of continuity of which,

$$\omega(f, \delta)_C = \sup_{|h| \leqslant \delta} \|f(\cdot + h) - f(\cdot)\|_C,$$

we have

$$\omega(f, \delta)_C = O(\omega(\delta)).$$

It is well known (see e.g. [5, p. 176]) that the sequence λ is of the type (C, C) if and only if

$$\sum_{k=-\infty}^{\infty} \lambda_k e^{ikx} \tag{3}$$

is a Fourier–Stieltjes series. For the type (C_ω, C_ω) this condition is sufficient but no longer necessary.

In 1959 A. Zygmund [4] proved that if $\omega(\delta) = \delta^\alpha$ with $0 < \alpha < 1$, then $\lambda \in (C_\omega, C_\omega)$ if and only if the indefinite integral of the series (3),

$$\mathscr{L}(x) = \sum_{k \neq 0} (ik)^{-1} \lambda_k e^{ikx}, \tag{4}$$

belongs to the Zygmund class in integral metrics L_*, i.e., $\omega_2(\mathscr{L}, \delta)_L = \sup_{|h| \leqslant \delta} \|\mathscr{L}(\cdot + 2h) + \mathscr{L}(\cdot) - 2\mathscr{L}(\cdot + h)\|_L = O(\delta)$.

In [1] it is shown that this theorem is valid for a wider class of moduli of continuity, namely for those satisfying the condition

$$\int_0^\delta \frac{\omega(t)}{t}\, dt + \delta \int_\delta^{2\pi} \frac{\omega(t)}{t^2}\, dt = O(\omega(\delta)). \tag{5}$$

(See also [2].) More precisely

THEOREM A. *A necessary condition for λ to be of the type (C_ω, C_ω) is that $\mathscr{L}$ should belong to L_*. If the modulus of continuity $\omega(\delta)$ is such that* (5) *holds, then this condition is also sufficient.*

In the present paper, we shall find a necessary and sufficient condition for λ to be of the type (C_ω, C_ω) with $\omega(\delta)$ slowly decreasing. We shall also show that if $\omega(\delta)$ satisfies (5) then this condition is equivalent to Theorem A.

2. PRELIMINARIES

Let $E_n(f)_L$ denote the best approximation of the function f with trigonometric polynomials of a degree not exceeding n in integral metrics. Let $v_n f$ denote the n, $2n$ de la Valleé Poussin means of the Fourier series of f ($s_n f$ is the nth symmetrical partial sum of (1))

$$v_n f = \left(\frac{1}{n}\right)(s_n f + s_{n+1} f \cdots + s_{2n-1} f)$$

and let $v_n(x)$ be the corresponding kernel

$$v_n(x) = 2K_{2n-1}(x) - K_{n-1}(x),$$

where

$$K_n(x) = \sum_{k=-n}^{n} (1 - |k|/n) e^{ikx}$$

is the Fejér kernel. By $f * g$ we denote the convolution

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, g(t)\, dt.$$

For any complex number z define the function $\overline{\text{sign}}\, z = \bar{z}/z$ for $z \neq 0$, $\overline{\text{sign}}\, z = 0$ if $z = 0$. By $v_n \Lambda$ we denote the de la Valleé Poussin means of the series (3).

PROPOSITION 1. *A necessary and sufficient condition for $\mathscr{L}$ to belong to the Zygmund class L_* is that for arbitrary $m, n \to \infty$, $m > n$, $m/n = O(1)$, we should have*

$$\|v_m \Lambda - v_n \Lambda\|_L = O(1). \tag{6}$$

Proof. Suppose $\mathscr{L} \in L_*$. That implies $E_n(\mathscr{L})_L = O(1/n)$. Considering Bernstein's inequality and the properties of the de la Valleé Poussin means we get

$$\begin{aligned}
\|v_m \Lambda - v_n \Lambda\|_L &\leqslant 2m \|v_m \mathscr{L} - v_n \mathscr{L}\|_L \\
&\leqslant 2m(\|v_n \mathscr{L} - \mathscr{L}\|_L + \|\mathscr{L} - v_n \mathscr{L}\|_L) \\
&\leqslant 8m(E_m(\mathscr{L})_L + E_n(\mathscr{L})_L) = O(m(1/m + 1/n)) \\
&= O(1 + m/n).
\end{aligned}$$

As we have supposed $m/n = O(1)$ this proves the necessity of (6). To prove sufficiency suppose that (6) holds. Then using Jackson's theorem we obtain the estimate

$$\begin{aligned}\|v_m\mathscr{L} - v_n\mathscr{L}\|_L &\leqslant 4E_n(v_m\mathscr{L})_L \leqslant (4/n)\, E_n(v_m\Lambda)_L \\ &\leqslant (4/n)\, \|v_m\Lambda - v_{[n/2]}\Lambda\|_L = O(1/n). \qquad (7)\end{aligned}$$

To estimate $E_n(\mathscr{L})_L$ consider the decomposition

$$\begin{aligned}\|\mathscr{L} - v_n\mathscr{L}\|_L &= \left\| \sum_{k=0}^{\infty} \{v_{2^{k+1}n}\mathscr{L} - v_{2^k n}\mathscr{L}\} \right\|_L \\ &\leqslant \sum_{k=0}^{\infty} \|v_{2^{k+1}n}\mathscr{L} - v_{2^k n}\mathscr{L}\|_L.\end{aligned}$$

In view of (7) this leads to

$$\|\mathscr{L} - v_n\mathscr{L}\|_L \leqslant \sum_{k=0}^{\infty} O\left(\frac{1}{2^k n}\right) = O(1/n),$$

which in its turn yields $E_n(\mathscr{L})_L = O(1/n)$, i.e., $\mathscr{L} \in L_*$. This concludes the proof of the proposition.

3. Main Theorem

Let $\omega(\delta)$ be a modulus of continuity. Consider the sequence $\Delta(\omega) = \{\delta_k\}_{k=0}^{\infty}$, defined by induction $(k \geqslant 0)$,

$$\begin{aligned}\delta_0 &= 2\pi, \\ \delta_{k+1} &= \min\left\{\delta : \max\left(\frac{\omega(\delta)}{\omega(\delta_k)}, \frac{\delta\omega(\delta_k)}{\delta_k\omega(\delta)}\right) = \frac{1}{2}\right\}. \qquad (8)\end{aligned}$$

The sequence $\Delta(\omega)$ has among others the following properties (see [3]):

1. $(1/c)\, \omega(\delta) \leqslant \sum_{k=0}^{\infty} \omega(\delta_k) \min(1, \delta/\delta_k) \leqslant c\omega(\delta);$ (9)
2. $\delta_k/\delta_{k+1} = O(1)$ $(k \to \infty)$ if and only if $\omega(\delta)$ satisfies the condition (5). (10)

THEOREM 1. *Let $\omega(\delta)$ be a slowly decreasing modulus of continuity, i.e., $\omega(\delta)/\sqrt{\delta} \uparrow \infty$ $(\delta \to 0+)$. Let $\Delta(\omega) = \{\delta_k\}$ be defined by (8) and let*

$n_k=[1/\delta_k]$. *Then a necessary and sufficient condition for* $\lambda=\{\lambda_k\}$ *to be of the type* (C_ω, C_ω) *is*

$$\|v_{n_{k+1}}A-v_{n_k}A\|_L=O(1). \tag{11}$$

Proof. Sufficiency. Let $f\in C_\omega$ and suppose (11) holds. Consider the series ($m_k=[n_k/2]$)

$$\sum_{k=0}^{\infty}(v_{n_{k+1}}f_\lambda-v_{n_k}f_\lambda)$$
$$=\sum_{k=0}^{\infty}(v_{2n_{k+1}}f-v_{m_k}f)*(v_{n_{k+1}}A-v_{n_k}A). \tag{12}$$

Since $\|v_{2n_{k+1}}f-v_{m_k}f\|_C=O(\omega(1/m_k))=O(\omega(1/n_k))$ we see that by (11) and (8) the series (12) converges uniformly and thus defines a continuous function f_λ. Let us estimate the modulus of continuity of f_λ. We have for $0<h<\delta$

$$\|f_\lambda(x+h)-f_\lambda(x)\|_C=\|\Delta_h f_\lambda\|_C$$
$$=\left\|\sum_{k=0}^{\infty}\Delta_h\{v_{n_{k+1}}f_\lambda-v_{n_k}f_\lambda\}\right\|_C$$
$$=\left\|\sum_{k=0}^{\infty}\{\Delta_h f\}*\{v_{n_{k+1}}A-v_{n_k}A\}\right\|_C$$
$$=\left\|\sum_{k=0}^{\infty}\{\Delta_h(v_{2n_{k+1}}f-v_{m_k}f)\}*\{v_{n_{k+1}}A-v_{n_k}A\}\right\|_C$$
$$\leqslant\sum_{k=0}^{\infty}\|\Delta_h(v_{2n_{k+1}}f-v_{m_k}f)\|_C\,\|v_{n_{k+1}}A-v_{n_k}A\|_L.$$

Let us examine the norm of the difference $\Delta_h(v_{2n_{k+1}}f-v_{m_k}f)$. On the one hand we have by Jackson's theorem

$$\|\Delta_h(v_{2n_{k+1}}f-v_{m_k}f)\|_C\leqslant 2\,\|v_{2n_{k+1}}f-v_{m_k}f\|_C$$
$$=O(E_{m_k}(f)_C)=O(\omega(1/m_k))=O(\omega(\delta_k)). \tag{13}$$

On the other hand, applying Bernstein's inequality we obtain

$$\|\Delta_h(v_{2n_{k+1}}f-v_{m_k}f)\|_C=O(n_{k+1}h\,\|v_{2n_{k+1}}f-v_{m_k}f\|_C)$$
$$=O(\delta n_{k+1}\omega(\delta_k)).$$

As $\omega(\delta)$ is slowly decreasing, in view of (8) we get

$$\|\Delta_h(v_{2n_{k+1}}f - v_{m_k}f)\|_C = O(\omega(\delta_k)\delta/\delta_{k+1})$$
$$= O(\omega(\delta_{k+1})\delta/\delta_{k+1}). \tag{14}$$

Combining (13) and (14) we obtain by (9) and (11)

$$\|\Delta_h f_\lambda\|_C = O\left(\sum_{k=0}^{\infty} \omega(\delta_k)\min(1, \delta/\delta_k)\right) = O(\omega(\delta)),$$

thus $f_\lambda \in C_\omega$. This proves the sufficiency part of the theorem.

Necessity. In view of Theorem A we only have to consider the case $\mathscr{L} \in L_*$. Suppose (11) does not hold and that $\mathscr{L} \in L_*$. Then there exists a sequence of indices $\{k(l)\}$ such that

$$\|v_{n_{k(l)+1}}\Lambda - v_{n(k(l)}\Lambda\|_L \geqslant l. \tag{15}$$

Let

$$G_k(x) = \overline{\text{sign}}\{v_{n_{k+1}}\Lambda(-x) - v_{n_k}\Lambda(-x)\}$$

and

$$\phi_k(x) = G_k(x) * \{v_{n_{k+1}}(x) - v_{n_k}(x)\}.$$

The functions ϕ_k are trigonometric polynomials of the order $2n_{k+1}$, orthogonal to trigonometric polynomials of an order less than n_k. Let

$$f(x) = \sum_{k=1}^{\infty} \omega(\delta_k)\,\phi_k(x).$$

Since $\|\phi_k\|_C = O(1)$ this series, in view of (8), converges uniformly, hence $f \in C$. To estimaıte the modulus of continuity of f observe that by Bernstein's inequality

$$\omega(\phi_k, \delta) = O(\min\{1, n_{k+1}\delta\}) = O(\min\{1, \delta/\delta_{k+1}\}).$$

Applying (9) we see that $f \in C_\omega$.

Next let us prove that $f_\lambda \notin C_\omega$. If f_λ is to belong to C_ω we should have $(m_k = [n_k/2])$

$$\|v_{2n_{k+1}}f_\lambda - v_{m_k}f_\lambda\|_C \leqslant 4(E_{2n_{k+1}}(f)_C + E_{m_k}(f_\lambda)_C)$$
$$= O(\omega(1/n_k)) = O(\omega(\delta_k)). \tag{16}$$

On the other hand we have chosen the polynomials ϕ_k so that

$$v_{2n_{k+1}} f_\lambda - v_{m_k} f_\lambda = \{\omega(\delta_{k-1})\phi_{k-1} + \omega(\delta_k)\phi_k + \omega(\delta_{k+1})\phi_{k+1}\} * \{v_{2n_{k+1}} \Lambda - v_{m_k} \Lambda\}.$$

Observing that all terms in the last convolution are trigonometric polynomials of specially chosen order we obtain

$$\begin{aligned} v_{2n_{k+1}} f_\lambda - v_{m_k} f_\lambda &= \omega(\delta_k)(\phi_k)_\lambda + \omega(\delta_{k-1}) G_{k-1} * \{v_{n_k} \Lambda - v_{m_k} \Lambda\} \\ &\quad + \omega(\delta_{k+1}) G_{k+1} * \{v_{2n_{k+1}} \Lambda - v_{n_{k+1}} \Lambda\}. \end{aligned}$$

Since $\|G_k\|_C \leqslant 1$ the application of Young's inequality gives us

$$\begin{aligned} \|v_{2n_{k+1}} f_\lambda - v_{m_k} f_\lambda\|_C &\geqslant \omega(\delta_k)(\phi_k)_\lambda(0) - \omega(\delta_{k-1}) \|v_{n_k} \Lambda - v_{m_k} \Lambda\|_L \\ &\quad - \omega(\delta_{k+1}) \|v_{2n_{k+1}} \Lambda - v_{n_{k+1}} \Lambda\|_L. \end{aligned}$$

In view of Proposition 1 and (8) this gives us

$$\|v_{2n_{k+1}} f_\lambda - v_{m_k} f_\lambda\|_C \geqslant \omega(\delta_k)(\phi_k)_\lambda(0) + O(\omega(\delta_k)).$$

Using the construction of ϕ_k we obtain by virtue of (15) ($k = k(l)$)

$$\begin{aligned} (\phi_{k(l)})_\lambda(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\operatorname{sign}}\{v_{n_{k+1}} \Lambda(-t) - v_{n_k} \Lambda(-t)\} \\ &\quad * \{v_{n_{k+1}} \Lambda(-t) - v_{n_k} \Lambda(-t)\}\, dt \\ &= \|v_{n_{k+1}} \Lambda - v_{n_k} \Lambda\|_L \geqslant l. \end{aligned}$$

Thus

$$\begin{aligned} \|v_{2n_{k(l)+1}} f_\lambda - v_{m_{k(l)}} f_\lambda\|_C &\geqslant \omega(\delta_{k(l)}) l + O(\omega(\delta_{k(l)})) \\ &\neq O(\omega(\delta_{k(l)})). \end{aligned}$$

which contradicts (16). This concludes the proof of the theorem.

Remark. If $\omega(\delta)$ satisfies condition (5) then in view of (10) and Proposition 1 this theorem becomes equivalent to Theorem A.

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